

UNIT-IPROBABILITY AND RANDOM VARIABLES.Conditional probability

The probability of the outcome  $B_j$  given that  $A_i$  is known as conditional probability. It is also called as transition probability.

If  $N_A$  represents the number of times an event  $A$  happens,  $N_B$  represents the number of times event  $B$  happens and  $N_{AB}$  represents the number of times of joint happening, then

$$P\left[\frac{A}{B}\right] = \frac{N_{AB}}{N_A} = \frac{N_{AB}/N}{N_A/N} = \frac{P(AB)}{P(A)}, \text{ for } P(A) > 0.$$

Similarly  $P\left[\frac{A}{B}\right] = \frac{N_{AB}}{N_B} = \frac{N_{AB}/N}{N_B/N} = \frac{P(AB)}{P(B)}, \text{ for } P(B) > 0.$

Example:-

Consider a bag of consisting of 100 items of which 80 are non defective and 20 are defective. Suppose that we have to choose two items from this. Let (a) with replacement (b) without replacement.

Let the events be  $A = \{\text{first item is defective}\}$

and  $B = \{\text{second item is defective}\}$ .

If the choosing is with replacement,

$$P(A) = P(B) = \frac{20}{100} = \frac{1}{5}$$

Among the total probability of 100

let the choosing be without replacement

then,  $P(A) = \frac{20}{100} = \frac{1}{5}$

Now, if the event  $A$  has occurred, then on the second drawing, there are only 99 items left, of which 19 are

defective.

Now, the probability that if the event A has occurred, then on the second drawing, there are only 99 items left, of which 19 are defective.

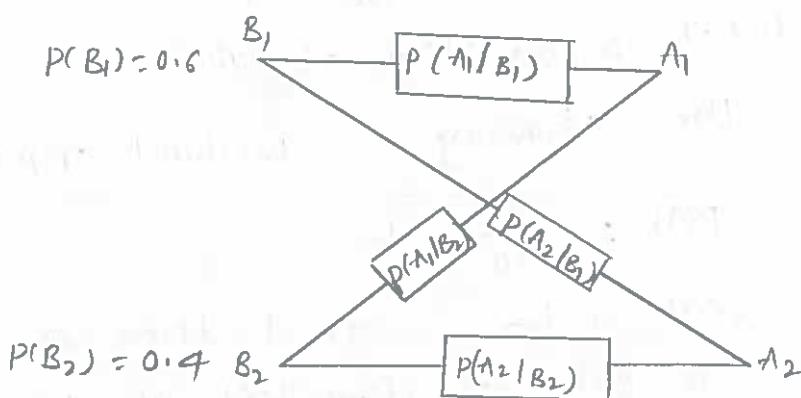
Now, the probability of B is denoted by the conditional probability  $P(B|A)$ , the condition being that the event A has occurred.

$$\therefore P(B|A) = \frac{19}{99}$$

Whenever, we compute  $P(B|A)$ , essentially  $p(B)$  is being computed with respect to the reduced sample space, rather than with respect to the original sample space.

$P(B|A)$  can be read as "probability of B given A".

Problem: Determine probabilities of error of system and correct system transmission of symbols for an elementary binary communication system shown in fig 1. consisting of a transmitter that sends one of two possible symbols (1) over a channel to a receiver. The channel occasionally causes errors to occur so that a '1' show up at the receiver as a '0' and vice versa. Assume the symbols '1' and '0' are selected for a transmission as 0.6 and 0.4 respectively.



The effect of the channel on the transmitted symbols is described by conditional probabilities.

Let us assume the reception probabilities given  $a_1$  was transmitted to be,

$$P(A_1|B_1) = 0.9, \quad P(A_2|B_1) = 0.1$$

We pre-assume the channel effects '0' in the same manner

$$P(A_1|B_2) = 0.1 \quad P(A_2|B_2) = 0.9$$

As seen in both cases

$$P(A_1|B_i) + P(A_2|B_i) = 1$$

Since  $A_1$  and  $A_2$  are mutually exclusive and are the only receiver events possible.

From the theorem of total probability

$$P(A) = \sum_{n=1}^N P(A|B_n) P(B_n) \quad [\text{Where } \sum_{n=1}^N B_n = s,$$

$N$  = total no. of events in  $s$

By using above theorem we obtain probabilities of  $A_1$  and  $A_2$  i.e received symbol probabilities as,

$$P(A_1) = P(A_1|B_1) P(B_1) + P(A_1|B_2) P(B_2) = 0.9(0.6) + 0.1(0.4) = 0.58$$

$$P(A_2) = P(A_2|B_1) P(B_1) + P(A_2|B_2) P(B_2) = 0.1(0.6) + 0.9(0.4) = 0.2$$

From Bayes theorem

$$P(B_n|A) = \frac{P(A|B_n) P(B_n)}{P(A)}$$

Let  $P(A_1|B_1)$  and  $P(B_2|A_2)$  represent the probability of correct system transmission of symbols and are obtained by using the above Baye's form.

$$\text{i.e. } P(B_1|A_1) = \frac{P(A_1|B_1) P(B_1)}{P(A_1)} = \frac{0.9(0.6)}{0.58} = 0.931.$$

$$P(B_2|A_2) = \frac{P(A_2|B_2) P(B_2)}{P(A_2)} = \frac{0.9(0.4)}{0.42} = 0.857.$$

Let  $P(B_1|A_2)$  and  $P(B_2|A_1)$  are the probabilities of system error then

$$P(B_1|A_2) = \frac{P(A_2|B_1) P(B_1)}{P(A_2)} = \frac{0.1(0.6)}{0.42} = 0.143.$$

$$P(B_2|A_1) = \frac{P(A_1|B_2) P(B_2)}{P(A_1)} = \frac{0.1(0.4)}{0.58} = 0.069.$$

$\therefore$  The probabilities of system error are  $P(B_1|A_2)$  and  $P(B_2|A_1)$  are 0.143 and 0.069 respectively.

Total probability :-

It can be stated as

"If the probability of an event happening as a result of a trial is  $P(A)$  and the probability of a mutually exclusive event  $B$  happening as a result of the trial is  $P(A+B)$  or  $P(A \cup B) = P(A) + P(B)$ ".

Proof :-

Let  $i$  be the total number of equally likely cases and let  $J_1$  be favourable to the event  $A$  and  $J_2$  be favourable to the event  $B$ .

Then the number of cases favourable to  $A$  or  $B$  is  $J_1+J_2$ .

Hence, the probability of  $A$  or  $B$  happening, as a result of the trial is

$$= \frac{J_1+J_2}{i} = \frac{J_1}{i} + \frac{J_2}{i} = P(A) + P(B).$$

If the events  $A$  and  $B$  are not mutually exclusive, then, there are some outcomes which favour both  $A$  and  $B$ . If  $J_3$  be their number, then these

are included in both  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

Hence, the total number of outcomes favouring either A or B or both is

$$\tau_1 + \tau_2 - \tau_3$$

Thus, the probability  $P(A+B)$  or  $P(A \cup B)$  of occurrence of A and B or both

$$= \frac{J_1 + J_2 - J_3}{i} = \frac{J_1}{i} + \frac{J_2}{i} - \frac{J_3}{i}.$$

$$\therefore P(A+B) = P(A) + P(B) - P(AB)$$

$$(01) \quad P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive

$$P(A \cap B) = P(AB) = 0$$

$$P(A+B) = P(A \cup B) = P(A) + P(B)$$

## Problem

Urn A contains 5 red marbles and 3 white marbles. Urn B contains 2 red marbles and 6 white marbles.

(a) If a marble is drawn from each urn, what is the probability that they are both of the same colour?

(b) If two marbles are drawn from each urn, what is the probability that all four marbles are of the same colour?

(a) P (selecting a red Marble from urn A) =  $\frac{5}{8}$ .

$$P(\text{selecting a white marble from Vin A}) = 3/8$$

P(C Selecting a red Marble, from Urn B) = 2/8

$P(\text{selecting a white marble from Urn B}) = 6/8$

$$P(\text{Both Marbles are of same colour}) = P(\text{Both are red}) + \\ \text{with } 1 \text{ red} \quad P(\text{Both are white})$$

$$= \frac{5}{8} \times \frac{2}{8} + \frac{3}{8} \times \frac{6}{8} = \frac{28}{64} = \frac{7}{16}.$$

(b). p (selecting 2 red Marbles from Urn A) =  $\frac{5C_2}{8C_2}$

p (selecting 2 white marbles from Urn A) =  $\frac{3C_2}{8C_2}$

p (selecting 2 red Marbles from Urn B) =  $\frac{2C_2}{8C_2}$

p (selecting 2 white marble from Urn B) =  $\frac{6C_2}{8C_2}$

P(All the 4 are of same colour):

$$\frac{5C_2}{8C_2} \cdot \frac{2C_2}{8C_2} + \frac{3C_2}{8C_2} \cdot \frac{6C_2}{8C_2} = \frac{55}{784}.$$

### Bayes's Theorem

Let  $B_1, B_2, \dots, B_n$  are mutually exclusive events whose union is the sample space is  $S$ , i.e.  $B_1, B_2, \dots, B_n$  are the partitions of the sample space is  $S$ , i.e. at least one of the events must occur.

Then if  $A$  is any event or an event  $A$  corresponds to a number of exhaustive events  $B_1, B_2, \dots, B_n$ .

If  $p(B_i)$  and  $p\left(\frac{A}{B_i}\right)$  are given,

$$P\left[\frac{B_i}{A}\right] = \frac{p(B_i) \cdot P\left(\frac{A}{B_i}\right)}{\sum_i p(B_i) \cdot P\left(\frac{A}{B_i}\right)}$$

where  $i = 1, 2, \dots, n$ .

This result is known as Baye's theorem.

4.

This enables us to find the probabilities of the various events  $B_1, B_2, \dots, B_n$  that can cause A to occur.

Since  $B_i$ 's are a partition of sample space, only one of the events  $B_i$  occurs, i.e. one of the events  $B_i$  must occur and only one can occur.

Hence, the above expression gives the probability of a particular  $B_i$ , i.e. a cause, given that the event A has occurred.

Hence, Baye's theorem is also called the formula for the probability of causes or theorem on the probability of causes.

Proof:-

By the law of multiplication of probability,

$$P(A \cap B_i) = P(A|B_i) = P(A) \cdot P\left(\frac{B_i}{A}\right) = P(B_i) \cdot P\left(\frac{A}{B_i}\right)$$

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i) \cdot P\left(\frac{A}{B_i}\right)}{P(A)}$$

since the event A corresponds to  $B_1, B_2, \dots, B_n$ , by using the theorem of total probability.

$$P(A) = \sum_i P(B_i) \cdot P\left(\frac{A}{B_i}\right)$$

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i) \cdot P\left(\frac{A}{B_i}\right)}{\sum_i P(B_i) \cdot P\left(\frac{A}{B_i}\right)}$$

The probabilities  $P(B_i)$ ,  $i=1, 2, \dots, n$  are called a prior probabilities, since these exist before we get any information from the experiment.

The probabilities  $P\left(\frac{A}{B_i}\right)$ ,  $i = 1, 2, \dots, n$  are called posteriori probabilities, because, these are found after the experiment results are known.

Problem :

State and prove Baye's theorem of probability.

We know that conditional probability of event B can be given by

$$P\left(\frac{B}{A}\right) = \lim_{n \rightarrow \infty} \frac{n_{AB}}{n_A} = \lim_{n \rightarrow \infty} \left[ \frac{n_{AB}/n}{n_A/n} \right] = \frac{P(A, B)}{P(A)}.$$

$$\Rightarrow P(A, B) = P\left(\frac{B}{A}\right) P(A)$$

Similarly,  $P\left(\frac{A}{B}\right) = \frac{P(A, B)}{P(B)}$

$$\Rightarrow P(A, B) = P\left(\frac{A}{B}\right) P(B)$$

Equations ① + ② represent Baye's rule.

Statement of Baye's Theorem:-

The joint probability of two events may be expressed as the product of the conditional probability of one event, given the other, times the elementary probability of the other.

Sample space:-

The set of all possible outcomes of a random experiment is called as a 'sample space' and each constituent outcome is called sample point.

Example:- The set of possible outcomes in the experiment of tossing a die is  $S = \{1, 2, 3, 4, 5, 6\}$ . which is a sample space.

5

Another Sample space for the above experiment is

$$S = \{\text{odd}, \text{even}\}$$

Problem:-

Define and Explain the following with example.

1. Sample space
- 2) Discrete sample space
- 3) Continuous sample space.

Sample space:- A set of all possible distinct of an random experiment is known as sample space. It is denoted by 'S'.

2. Continuous sample space: If the sample space contains uncountable infinite number of events, then the sample space is called as continuous sample space.

Example:- Consider an experiment of measuring room temperature from  $t_1$  to  $t_2$  (sec).

Here sample space  $S = (t_1 < s < t_2)$

Such a sample space is called continuous.

3. Discrete sample space:- If sample space contains finite set of events then it is said to be discrete sample space.

Example:- a. In tossing a coin, sample space  $S = \{H, T\}$ .

b. In rolling a die,  $S = \{1, 2, 3, 4, 5, 6\}$ .

As seen from the example of the discrete and finite.

